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THE SIMPLEST DISCRETE SYSTEM WITH RELAXATION†

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The existence and stability of the positive periodic modes of a system, described by the scalar equation

$$x_n = \begin{cases} x_{n-1} - hx_{n-2}; & x_{n-1} - hx_{n-2} > 0 \\ 1; & x_{n-1} - hx_{n-2} \le 0 \end{cases}$$

are investigated for $n \in \mathbb{N} \cup \{0\}$ and different $h = \text{const} > 0$.

1. INTRODUCTION

In [1] we considered a *continuous* system with delay and relaxation, described by the scalar equation

$$\dot{x}(t) = f(x(t+\theta_1), \dots, x(t+\theta_m), x_t) \tag{1.1}$$

where θ_i are specified non-positive constants, $x_t(\theta) := x(t+\theta)(\theta_0 \le \theta \le 0)$, and the relaxation condition

$$x(t) = 0 \Rightarrow x(t^+) = l, \quad l = \text{const} > 0 \tag{1.2}$$

The sufficient conditions were obtained such that this system has exactly one periodic solution (apart from arbitrary shifts along the t axis), to which any non-negative solution of the same problem (1.1), (1.2) tends as $t \to \infty$. A similar result was obtained in [2] for $m = \infty$, $\theta_0 = -\infty$.

The purpose of this paper is to investigate the simplest *discrete* system with relaxation, described by the scalar equation

$$x_n = x_{n-1} - hx_{n-2}; \quad n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{0\}, \quad h = \text{const} > 0 \tag{1.3}$$

We will assume the specified initial quantities x_{-2} and x_{-1}, and all the quantities $x_n (n \in \overline{\mathbb{N}})$ to be positive (this requirement will not be specifically stated henceforth). The relaxation condition will be taken as follows: if, in a successive calculation of $x_0, x_1, \dots$ using formula (1.3), we first obtain $x_{n_1} \le 0$, then, instead of this, we assume $x_{n_1} = 1$, and when $n > n_1$ we construct a solution of Eq. (3) using $x_{n_1 - 1}$ obtained and $x_{n_1} = 1$ as the initial values, up to the next value $x_{n_2} \le 0$, instead of which we assume $x_{n_2} = 1$, etc. In other words, Eq. (1.3) with the relaxation condition can be written as

$$x_{n-1} - hx_{n-2} = \begin{cases} x_n & \text{for} \quad x_{n-1} - hx_{n-2} > 0 \\ 1 & \text{for} \quad x_{n-1} - hx_{n-2} \le 0 \end{cases}, \quad n \in \overline{\mathbb{N}}$$

We similarly determine the solution of Eq. (1.3) with the relaxation condition for $n \ge n_0$, and also for $n \in \mathbb{Z}$. We will call the problem of constructing a solution of Eq. (1.3) with the relaxation condition Problem R.

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2. PERIODIC SOLUTIONS OF SMALL PERIOD

Periodic solutions $\{x_n\}$ of Problem R will be considered for $n \in \mathbb{Z}$. These solutions can be classified in accordance with the length N of their least period. For periodic solutions we always have $\max_n\{x_n\} = 1$, and hence, without loss of generality, we can assume that $x_1 = 1$. The sequence $\{x_1, \dots, x_N\}$ will be called a *cycle* of the periodic solution considered. It is obvious that each cycle consists of a certain number $p \geq 1$ of *fragments*, following one another, each of which is a decreasing sequence, beginning with 1.

When seeking the form of the cycles of given length, the following simple theorem is useful.

Theorem 1. For $N > 1$ all the fragments of a cycle have a length greater than 1. For $N > 4$ at least one fragment has a length greater than 2.

Proof. Suppose $N > 1$ and, to be specific, the fragment of the cycle has a length 1, i.e. $x_2 = 1$. We will put

$$k := \min\{n \in \{1, \dots, N\} : x_n < 1\}$$

We have $1 - hx_N \leq 0, x_k = 1 - h$. But it follows from the equality $x_k = 1 - h$ that $h < 1$, which contradicts the inequality $1 - hx_N \leq 0$.

Suppose now that $N > 4$ and all the fragments of the cycle have a length 2. Then N is an even number, and we have

$$1 - hx_{2i-2} = x_{2i} \tag{2.1}$$

$$x_{2i} - h \leq 0 \tag{2.2}$$

(Here and henceforth in this section $i = 1, \dots, N/2; x_0 = x_N$). By virtue of its cyclical character it follows from the system of equalities (2.1) that

$$x_{2i} = \sum_{j=0}^{(N/2)-1} (-h)^j + (-h)^{N/2} x_{2i}$$

Hence we see that if the number $N/2$ is odd, all the numbers x_{2i} are equal, and hence the least period is 2, and not N. We obtain the same contradiction if the number $N/2$ is even and $h \neq 1$. Finally, if the number $N/2$ is even and $h = 1$, it follows from the equality $x_{2i-2} + x_{2i} = 1$ that all the numbers x_{4i} are equal, like all the numbers x_{4i+2}, and hence the least period is 4 and not N.

We will present the form of the cycles for the small values of N.

The case $N = 1$. The cycle has the form $\{1\}$. A periodic solution with $N = 1$ is possible if and only if $h \geq 1$.

The case $N = 2$. The cycle has the form $\{1, a\}$, where $a \in (0, 1)$. The relations $a = 1 - ha$, $a - h \leq 0$ are obvious. Representing the line and the region corresponding to these relations in the a, h plane, we obtain the necessary and sufficient condition for periodic solutions to exist with $N = 2 : h \geq \sqrt{5/4} - 1/2 \approx 0.618$, and the cycle has the form

$$\{1, 1/(1 + h)\}$$

It is obvious that when $h \geq 1$ periodic solutions are possible both with $N = 1$ and with $N = 2$.

The case $N = 3$. By virtue of Theorem 1, here a cycle has the form $\{1, a, b\}$. In this case the relations $a = 1 - hb > 0, b = a - h > 0, b - ha \leq 0$ must be satisfied. Eliminating b from them we arrive at the relations $a = 1 - ha - h^2, 0 < a < 1, a - h > 0, a - h - ah \leq 0$. Representing the corresponding line and regions in the a, h plane, we obtain that a periodic solution with $N = 3$ is possible if and only if $h_1 \leq h < 1$, where $h_1 \approx 0.453$ is the only real root of the equation $h^3 + 2h - 1 = 0$. In this case the cycle has the form

$$\{1, (1 + h^2)/(1 + h), (1 - h)/(1 + h)\}$$

Hence, when $\sqrt{5/4} - 1/2 \leq h < 1$ periodic solutions are possible both with $N = 2$ and $N = 3$.

The case $N = 4$. Here, by virtue of Theorem 1, the cycle can have the form $\{1, a, b, c\}$ $(1 > a > b > c > 0)$ or $\{1, a, 1, b\}$ $(1 > a > b > 0)$. We can conclude, similar to the previous case, that a cycle of the first form is possible if and only if $h_2 \le h < 1/2$, where $h_2 \approx 0.373$ is the only positive root of the equation $h^4 - h^2 + 3h - 1 = 0$. In this case the cycle has the form

$$\{1, (1+h^2)/(1+h-h^2), (1-h+h^3)/(1+h-h^2), (1-2h)/(1+h-h^2)\}$$

When $h_1 \le h < 1/2$ periodic solutions are possible both with $N = 3$ and when $N = 4$ (of the first form).

For cycles of the second form, it follows from the proof of Theorem 1 that $h = 1$, and the cycle has the form $\{1, a, 1, 1-a\}$ with any $a \in (1/2, 1)$. Hence, when $h = 1$ periodic solutions are possible with $N = 1$, $N = 2$ and $N = 4$.

The case $N = 5$. It follows from Theorem 1 that the cycle can have the form $\{1, a, b, c, d\}$ $(1 > a > b > c > d > 0)$ or $\{1, a, 1, b, c\}$ $(1 > a > 0, 1 > b > c > 0)$. We conclude, as in the previous case, that a cycle of the first form is possible if and only if $h_3 \le h < 3/2 - \sqrt{5/4} \approx 0.382$, where $h_3 \approx 0.331$ is the only positive root of the equation $h^5 - 3h^2 + 4h - 1 = 0$. In this case the cycle has the form

$$\{1, (1+h^2-h^3)/(1+h-2h^2), (1-h+h^3)/(1+h-2h^2),$$
$$(1-2h+h^4)/(1+h-2h^2), (1-3h+h^2)/(1+h-2h^2)\}$$

Hence, when $h_2 \le h < 3/2 - \sqrt{5/4}$ periodic solutions are possible both with $N = 4$ (of the first form) and with $N = 5$ (also of the first form).

For cycles of the second form, after eliminating c and b, we obtain the relations

$$a(1-h^2) = 1-h+h^2, \quad a \le h, \quad ah < 1-h, \quad a(h-h^2) \ge 1-2h$$

One can verify that the first three of these lead to a contradiction. Hence, there are no periodic solutions with cycles of the second form.

The case $N = 6$. It follows from Theorem 1, that the cycle can have the form $\{1, a, b, c, d, e\}$ $(1 > a > b > c > d > e > 0)$, or $\{1, a, 1, b, c, d\}$ $(1 > a > 0, 1 > b > c > d > 0)$, or $\{1, a, b, 1, c, d\}$ $(1 > a > b > 0, 1 > c > d > 0)$. We can conclude, as previously, that cycles of the first form are possible if and only if $h_4 \le h < 1/3$, where $h_4 \approx 0.307$ is the least of the two positive roots of the equation $h^5 + h^4 + h^3 + 2h^2 - 4h + 1 = 0$. In this case the cycle has the form

$$\{1, (1+h+2h^2)/(1+2h-h^2), (1+h^3)/(1+2h-h^2),$$
$$(1-h-h^2-h^3)/(1+2h-h^2), (1-2h-h^2-h^3-h^4)/(1+2h-h^2), (1-3h)/(1+2h-h^2)\}$$

Hence, when $h_3 \le h < 1/3$ periodic solutions are possible both with $N = 5$ and with $N = 6$ (of the first form).

For cycles of the second form, after eliminating d, c and b, we arrive at relations, the first three of which – $a(1-h^2+h^3) = 1-h+2h^2$, $a \le h$, $ah < 1-h$ – contradict one another. Hence, there are no cycles of the second form.

For cycles of the third form, eliminating d, c and b, we arrive at the relations

$$a(1-h^2) = 1-h+h^2-h^3, \quad a > h, \quad a(1-h) \le h, \quad ah < 1-h+h^2$$
$$a(h-h^2) \ge 1-2h+h^2-h^3$$

It follows from the second and fourth relations that $h \ne 1$, and hence the first can be divided by $1-h$. Expressing a from the equality obtained, and from this, b, c and d also, we arrive at the expressions

$$a = c = (1+h^2)/(1+h), \quad b = d = (1-h)/(1+h)$$

Hence, the cycles consists of two similar fragments, which is impossible. Thus there are no cycles of the third form.

From the results obtained we can naturally state the hypothesis that when $N \geq 2$ the values of h for which cycles of length N, consisting of a unique fragment, are possible, form the interval $[\alpha_N, \beta_N)$, where $\alpha_N < \beta_{N+1} < \alpha_{N-1} < \beta_N$ $(\forall N \geq 3)$ and $\alpha_N \to 1/4$ when $N \to \infty$ (see Section 3). The following simple assertion may turn out to be useful when proving this: successive elements of the cycle $\{1, a_1, a_2, \dots, a_{N-1}\}$ are proportional to the successive cofactors of the elements of the first row of the square matrix of order N, in which the second row has the form (-1, 1, $N-3$ times zero, h), while each successive row is obtained from the previous one by cyclical permutation by putting the last element in the first position. For example, when $N = 4$ the last three rows of this matrix have the form

$$(-1, 1, 0, h),\ (h, -1, 1, 0),\ (0, h, -1, 1)$$

We do not know of any examples of cycles consisting of more than one fragment with $N \neq 4$.

3. THE CONDITION FOR THERE TO BE NO PERIODIC SOLUTIONS.

Theorem 2. When $h \leq 1/4$ Problem R has no periodic solutions.

Proof. Suppose the condition of Theorem 2 is satisfied, but Problem R has periodic solutions with the cycle $\{x_0 = 1, x_1, \dots, x_{N-1}\}$. Then, it follows from the results of Section 2 that $N \geq 7$. Suppose $N_0 := \min\{n \in \mathbb{N} : x_n = 1\}$. Then, by virtue of Theorem 1, we have $2 \leq N_0 \leq N$, where $N_0 \neq N - 1$.

We will first assume that $h < 1/4$. Then from the formula $x_n = C_1 p_1^n + C_2 p_2^n$ for the general solution of Eq. (1.3), where

$$p_1 = (1-\lambda)/2, \quad p_2 = (1+\lambda)/2, \quad \lambda = \sqrt{1-4h} \in (0, 1)$$

and C_1 and C_2 are arbitrary constants, taking into account the initial conditions $x_0 = 1, x_{-1} = x_{N-1} \in (0, 1)$, we obtain

$$x_n = (h/\lambda)[(x_{-1} - p_2^{-1})p_1^n - (x_{-1} - p_1^{-1})p_2^n], \quad n = -1, 0, \dots, N_0 - 1 \tag{3.1}$$

However, by the definition of N_0, the right-hand side of this expression when $n = N_0$ is non-positive. Hence, using simple reductions, we obtain the inequality

$$(1+\lambda)^{N_0}(1 + \lambda - 2hx_{-1}) \leq (1-\lambda)^{N_0}(1 - \lambda - 2hx_{-1}) \tag{3.2}$$

which is obviously untrue.

If $h = 1/4$, formulae (3.1) and (3.2) are replaced by $x_n = x_{-1}2^{-n-1} + (2 - x_{-1})(n+1)2^{-n-1}$ and $2(N_0 + 1) \leq x_{-1}N_0$ respectively, which leads to the same result.

As was noted in Section 2, it is possible that the condition $h \leq 1/4$ is not only sufficient but also necessary for there to be no periodic solutions of problem R.

4. THE STABILITY OF THE PERIODIC SOLUTIONS

The following simple assertion will be useful later.

Theorem 3. If $h \geq 1$, for periodic solutions of Problem R the least period N is equal to 1, 2 or 4.

Proof. Suppose $h \geq 1$ and $N \notin \{1, 2, 4\}$. It then follows the results to Section 2 that $N > 6$. If, in the corresponding cycle, some element, but not the last, is equal to 1, then, by virtue of Theorem 1, the next element is less than 1. If this element is not the last, the element following it is again equal to 1. Hence, the cycle consists of fragments of length 2, which is impossible in view of Theorem 1.

We will introduce in standard form the idea of stability, asymptotic stability and instability of the solution of Problem R with initial data regarding their changes. We will say that a periodic solution of Problem R is stable (asymptotically stable), if it is stable (correspondingly asymptotically stable) when there is a change in the initial data when any value of n is chosen as the initial value.

We will consider the stability of certain periodic solutions.

Suppose $N = 1$. It is then easy to verify that for $h > 1$, for any sufficiently small change in the initial data, the perturbed solution is identical with the unperturbed solution, which, therefore, is asymptotically stable. For $h = 1$ it is unstable; for example, when $x_{-2} = 1, x_{-1} = 1 - \varepsilon$ ($\varepsilon \in (0, 1)$) we have

$$x_{2n} = 1, \quad x_{4n+1} = \varepsilon, \quad x_{4n+3} = 1-\varepsilon, \quad \forall n \in \bar{\mathbb{N}}$$

Suppose $N = 2$. Then when $h > 1$ and when $h = \sqrt{5/4} - 1/2$ the periodic solution is unstable; when $h = 1$ it is non-asymptotically stable; when $\sqrt{5/4} - 1/2 < h < 1$ it is asymptotically stable. Indeed, let

$$h > \sqrt{5/4} - 1/2, \quad x_{-2} = 1 + \varepsilon_1, \quad x_{-1} = 1/(1+h) + \varepsilon_2$$

(see Section 2, $N = 2$). Then, for sufficiently small $|\varepsilon_1|$, $|\varepsilon_2|$ we have

$$x_0 = 1, \quad x_1 = 1/(1+h) - h\varepsilon_2, \quad x_2 = 1, \quad x_3 = 1/(1+h) + h^2\varepsilon_2$$

etc., so long as (and if) it is not required to use the relaxation condition. The initial conditions

$$x_{-2} = 1/(1+h) + \varepsilon_1, \quad x_{-1} = 1 + \varepsilon_2$$

lead to a similar sequence, whence we immediately obtain the assertion presented above regarding stability for the values of h considered. If $h = \sqrt{5/4} - 1/2$, then by choosing for the first form of the initial conditions values of ε_1 and ε_2 as small in modulus as desired, we can make the quantity x_0 positive and as small as desired, whence it follows that the periodic solution is unstable.

The above discussion has a general form. Bearing Theorem 3 in mind, as well as the result obtained in Section 2 for $N = 4$, we obtain the following assertion.

Theorem 4. Suppose $N \geq 3$, and the following condition is satisfied

$$x_n \neq hx_{n-1}, \quad \forall n = 1, \ldots, N$$

Then the periodic solution $\{x_n\}$ of Problem R is asymptotically stable. If this condition breaks down, this solution is unstable. In the special case when $N = 4$ and $h = 1$, the periodic solution is non-asymptotically stable.

It follows from this theorem and from the stated in Section 2, in particular, when $N \in [3, 6]$, that for the interval $h \in [\alpha_N, \beta_N)$ where a periodic solution exists this periodic solution is asymptotically stable when $h \in (\alpha_N, \beta_N)$ and unstable when $h = \alpha_N$.

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